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On Finding Integer Solutions on Binary Heptic Equation

$$x^2 - x y^3 = 2 y^7$$

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ABSTRACT

The non-homogeneous polynomial equation of degree five with two unknowns given by $x^2 - x y^3 = 2 y^7$ is studied to determine its distinct integer solutions . Some connections between the solutions are presented.

KEYWORDS : Binary heptic equation ,Non-homogeneous heptic equation ,Integer solutions

NOTATIONS

$$t_{3,n} = \frac{n\,(n+1)}{2}$$

$$P_n^5 = \frac{n^2 (n+1)}{2}$$

$$P_n^3 = \frac{n^3 + 3n^2 + 2n}{6}$$

$$S_n = 6n(n+1) + 1$$

$$Th_n = 3 * 2^n - 1$$

INTRODUCTION

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics.The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree Diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power Diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of Cubic and Quintic Diophantine equations with multi variables [1-27]. It seems that much work has not been done regarding polynomial Diophantine equations of degree seven with two unknowns. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree seven with two unknowns given by $x^2 - x y^3 = 2 y^7$. A few relations between the solutions are presented.

II.Method of analysis

The non-homogeneous quintic equation with two unknowns under consideration is

$$x^2 - x y^3 = 2 y^7 \quad (1)$$

Treating (1) as a quadratic in x and solving for the same ,we have

$$x = \frac{y^3 [1 \pm \sqrt{8\,y + 1}]}{2} \quad (2)$$

Consider the positive sign before the square-root in (2). After some calculations, it is seen that the square-root in (2) is removed when

$$y = y(s) = t_{3,s} \quad (3)$$

and correspondingly we have from (2)

$$x = x(s) = (s+1)\,(t_{3,s})^3 \quad (4)$$

Observe that (3)&(4) satisfy (1). A few numerical solutions to (1) are presented in Table-1 below:

Table-1:Numerical solutions

S	y=y(s)	x=x(s)
1	1	$2*1^3$
2	3	$3*3^3$
3	6	$4*6^3$
4	10	$5*10^3$
5	15	$6*15^3$
6	21	$7*21^3$
7	28	$8*28^3$

Relations observed:

1. $y(s+2) - 2\,y(s+1) + y(s) \equiv 1$
2. $y(s+1) - y(s) = S_a$ when $s = 6\,a\,(a+1)$
3. $x(s) = [y(s)]^3\,[y(s+1) - y(s)]$
4. $x(s) = [y(s)]^3\,[y(s+2) - y(s+1) - 1]$
5. $2\,x(s) = [y(s)]^3\,[y(s+2) - y(s) - 1]$
6. $y(s+2) - y(s+1)$ is a perfect square when $s = k^2 \mp 2\,k - 1$
7. $y(s+1) - y(s)$ is a perfect square when $s = k^2 \mp 2\,k$
8. $y(s+2) - y(s)$ is a perfect square when $s = 2\,k^2 \mp 2\,k - 1$
9. $2[y(s+2) + y(s+1) - y(s)] + 1 - s$ is a perfect square
10. $y(s+1) + y(s) = (s+1)^2$

Note 1

Taking the negative sign before the square-root in (2) and repeating the process as above , the corresponding integer solutions to (1) are given below:

$$y = y(s) = t_{3,s} \,, x = x(s) = -s(t_{3,s})^3$$

Remark 1

To remove the square-root in (2) , let

$$\alpha^2 = 8y + 1 \tag{5}$$

which , after some algebra , is satisfied by

$$y_0 = t_{3,s} \quad , \alpha_0 = 2s + 1 \tag{6}$$

Assume the second solution to (5) as

$$\alpha_1 = h - \alpha_0 \,, y_1 = h + y_0 \tag{7}$$

where h is an unknown to be determined. Substituting (7) in (5) and simplifying , we have

$$h = 2\alpha_0 + 8$$

and in view of (7) ,it is seen that

$$\alpha_1 = \alpha_0 + 8, y_1 = y_0 + 2\alpha_0 + 8$$

The repetition of the above process leads to the general solution to (5) as

$$\alpha_n = \alpha_0 + 8n = 2s + 1 + 8n\,,$$

$$y_n = y_n(s) = y_0 + 2n\alpha_0 + 8n^2 = \frac{1}{2}[(s + 4n)^2 + (s + 4n)] \tag{8}$$

Taking the positive sign before the square-root in (2) ,we get

$$x_n = x_n(s) = \frac{(y_n(s))^3 \; [1 + \alpha_n]}{2}$$

$$= (4n + s + 1)(y_n(s))^3 \tag{9}$$

Thus , the integer solutions to (1) are represented by (8)&(9).

Relations observed

1. $y_n(s+2) - 2y_n(s+1) + y_n(s) = 1$
2. $(y_n(s))^3 (y_n(s+1) - y_n(s)) = x_n(s)$
3. $(y_n(s))^3 (y_n(s+2) - y_n(s+1) - 1) = x_n(s)$
4. $(y_n(s))^3 (y_n(s+2) - y_n(s) - 1) = 2\, x_n(s)$
5. $\dfrac{x_n(s)}{(y_n(s))^3} = Th_n$ when $s = 3*2^n - 4n - 2$
6. $\dfrac{x_n(s)}{(y_n(s))^3} == S_n$ when $s = 6n^2 + 2n$
7. $\dfrac{x_n(s)}{(y_n(s))^3}$ is a perfect square when $s = n^2 + 3$
8. $(y_n(s))^3 (y_n(s) - y(s)) = 2n[s(y_n(s))^3 + x_n(s)]$

9. $2(y_n(s))^7 = x_n(s)[x_n(s)-(y_n(s))^3]$
10. $y_{n+2}(s)-2y_{n+1}(s)+y_n(s)=16$
11. $(y_n(s))^3(y_{n+1}(s)-y_n(s)-6)=4x_n(s)$
12. $(y_n(s))^3(y_{n+2}(s)-y_{n+1}(s)-22)=4x_n(s)$
13. $(y_n(s))^3(y_{n+2}(s)-y_n(s)-28)=8x_n(s)$
14. $(y_n(s))^3(y_{n+2}(s)-y_n(s+2)-27)=6x_n(s)$
15. $y_{n+2}(s)-y_n(s+2)-33=36P_n^3$ when $s=n^3+3n^2-2n$
16. $y_{n+2}(s)-y_n(s)-36=48P_n^3$ when $s=n^3+3n^2-2n$
17. $y_{n+2}(s)-y_n(s+2)-33=12P_n^5$ when $s=n^3+n^2-4n$
18. $y_{n+2}(s)-y_{n+1}(s)-26=8P_n^5$ when $s=n^3+n^2-4n$

Remark 2

Taking the negative sign before the square-root in (2) ,we get

$$x_n = \frac{(y_n(s))^3[1-\alpha_n]}{2}$$

$$= -(4n+s)(y_n(s))^3 \qquad (10)$$

Thus , the integer solutions to (1) are represented by (8)&(10).

III.Conclusion

The polynomial equation of degree seven with two unknowns given by $x^2 - xy^3 = 2y^7$ has been studied to obtain non-zero integer solutions .The process of eliminating the square-root will be beneficial for the researchers. As heptic equations are plenty , one may attempt to determine the solutions in integers for other choices of heptic Diophantine equations.

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